

A Lower Bound for the Mahler Volume of at Least Four-Dimensional Symmetric Convex Sets

Yashar Memarian

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Abstract

The goal of this paper is to present a lower bound for the Mahler volume of at least 4-dimensional symmetric convex bodies. I define a computable constant (depending on the dimension) through a 2-dimensional variational (max-min) procedure and demonstrate that the Mahler volume of every (at least 4-dimensional) symmetric convex body is greater than the product of this constant times the volume of the $(n-1)$ -dimensional canonical sphere to the power of two. Similar to the proof of Gromov's Waist of the Sphere Theorem in [18], my result is proved via localisation-type arguments obtained from a suitable measurable partition (or partitions) of the canonical sphere.

1 Introduction

The Mahler Conjecture is one of those amusing (and perhaps annoying) conjectures that have been floating around mathematical literature for decades. It originates (as the name indicates) from Mahler's studies at the end of the Thirties. It is one of those mathematical conjectures that everyone with a minimal knowledge in mathematics can understand. The idea is simple: consider (symmetric) convex sets in a fixed dimension and ask yourself what the *most* and *least* (symmetric) ones are. To be able to give a mathematical flavour to this question, Mahler introduced a *functional* on the class of (symmetric) convex sets, and was interested in minimising and maximising this functional. The functional is now known as the *Mahler volume* of a convex set. To understand this functional one should first know the definition of the polar of a (symmetric) convex body :

Definition 1.1 (Polar of a Convex Body) *Let K be a convex body in \mathbb{R}^n . The polar of K denoted by K° is the following:*

$$K^\circ = \{x \in \mathbb{R}^n \mid |x \cdot y| \leq 1, \forall y \in K\},$$

where \cdot stands for the inner product associated to the canonical Euclidean structure of \mathbb{R}^n .

It is not hard to verify that indeed K° is a convex body (should K be convex itself). Additionally, if we suppose K to be symmetric with respect to the origin of \mathbb{R}^n , K° satisfies the same property.

The Mahler volume of a (symmetric) convex body can be defined as:

$$M(K) = \text{vol}_n(K) \text{vol}_n(K^\circ),$$

and the interesting question is now to find:

$$\min_K M(K),$$

where K runs over the class of (symmetric) convex bodies.

Mahler raised this question in 1938 – 1939 in [33] and [34]. He succeeded to completely answer his own question for the 2-dimensional bodies. He realised that if one minimises the Mahler volume over the class of convex bodies, the minimum of the Mahler volume is attained for the simplex (i.e. the convex hull of three non-aligned points). When one seeks the minimum over symmetric convex bodies, the minimum is attained on the 2-dimensional cube (see [39] for another proof of the 2-dimensional Mahler question). After this discovery, he naturally generalised the question for higher dimensions. Here, I am only interested in cases where the convex bodies are supposed to be symmetric:

Conjecture 1.1 (Mahler Conjecture) *Let K be a symmetric convex body in \mathbb{R}^n , then*

$$\begin{aligned} \text{vol}_n(K)\text{vol}_n(K^\circ) &\geq \text{vol}_n(I^n)\text{vol}_n(I^{n^\circ}) \\ &= \frac{4^n}{\Gamma(n+1)}. \end{aligned}$$

where $\Gamma(n) = (n-1)!$ and is the (well-known) Gamma function :

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx,$$

and where I^n is the n -dimensional unit cube (i.e. $\text{vol}_n(I_n) = 1$).

Mahler did not just conjecture the above. He was also interested in an upper bound for the Mahler volume of (symmetric) convex bodies. He conjectured that the upper bound is achieved by the unit ball. The upper bound case for the Mahler question has been completely answered and is known as the Blaschke-Santaló inequality:

Theorem 1 (Blaschke-Santaló Inequality) *For every (symmetric) convex body K in \mathbb{R}^n where $n \geq 2$, we have:*

$$\begin{aligned} \text{vol}(K)\text{vol}(K^\circ) &\leq \text{vol}(B_n(0,1))\text{vol}(B_n(0,1)^\circ) \\ &= \text{vol}(B_n(0,1))^2 \\ &= \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \end{aligned}$$

where $B_n(0,1)$ is the n -dimensional unit ball in \mathbb{R}^n .

Theorem 1 (which was one part of Mahler's question) has been proved by Blaschke for the 3-dimensional case in [7] and [6] and in full generality by Santaló in [50]. There are also other proofs to this theorem : in [40] the authors prove this theorem using Steiner symmetrisation. In [49] the author proves that the equality case in Theorem 1 is obtained only for ellipsoids. In [27] the author proves a functional inequality version of this inequality using a partition-type method. One can also consult Tao's blog post [55] and [31] for more information on this inequality.

The lower bound for the Mahler volume of (symmetric) convex bodies however is still open for $n \geq 3$. Naturally, since Mahler raised Conjecture 1.1 many developments have been made

concerning this question. There are many partial results regarding this question which can be settled in different ways. One difficulty with Conjecture 1.1 is that the minimiser is not *unique*. Indeed the Mahler volume is invariant under affine transformations of \mathbb{R}^n . This complicates finding this minimiser. There is a very nice note about the complications within Conjecture 1.1 in [54]. Although one can (easily) deduce that a minimiser for the Mahler volume exists (Mahler himself was able to demonstrate this fact), not much was known about the *shape* of such a minimiser until very recently. In [53], the author demonstrates that the boundary of the minimiser can not be positively curved and of class C^2 everywhere. In [47], the authors prove a result suggesting that the minimiser of the Mahler volume is indeed a polytope.

Though the Mahler Conjecture suggests that the global minimiser for the Mahler volume is the cube, one could also seek the local minimiser of this functional. In [43], the authors demonstrate that the (unit) cube is indeed a local minimiser for the Mahler functional.

As proving Conjecture 1.1 in full generality has revealed many complications, a possible way to prove partial results related to this question would involve trying to prove the conjecture for more restricted (symmetric) convex bodies. In [49] and [46], the authors prove the conjecture if one restricts themselves to the class of unit balls of Banach spaces with a 1-unconditional basis. In [32] and [45], the author proves the conjecture for the Zonoids. A simplified proof of this fact can be found in [16]. In [5], the authors prove the conjecture for symmetric convex bodies with *many* symmetries. See also [29] for a partial result concerning Conjecture 1.1.

Another direction one could take to study Conjecture 1.1 would be by trying to find non-necessarily sharp lower bounds for the Mahler volume of (symmetric) convex bodies. Usually this is done by finding a constant $c(n)$ (depending on the dimension) such that the Mahler volume of *every* (symmetric) convex body is larger than $c(n)M(B_n(0,1))$. The most trivial constant one can easily obtain is by using the famous John's Lemma (see [23]) to obtain $c(n) = n^{-n/2}$. The first breakthrough is due to Bourgain-Milman in [8] where the authors prove the existence of a universal constant $c > 0$ such that $c(n) = c^n$. In [25] the author discovers another value for $c(n)$ and (to my knowledge), in [26] the same author proves the best known constant for $c(n)$.

Although Conjecture 1.1 can be categorised as a geometric problem, it has been demonstrated that much information about it can be obtained by using functional inequalities and analysis. For example, the Legendre transformation has had great importance in studying this conjecture (see [41], [13], [58] and [17]).

Numerous surveys and books on Convex Geometry (in which at least a few chapters are dedicated to the volume of polar bodies and/or the Mahler volume) can be consulted in: [4], [56], [44], [51], [20],[22] and [15].

The Mahler Conjecture has shown importance in other areas of mathematics as well. For example, in [3] the author shows the connection between Conjecture 1.1 to wavelets, and recently in [2] the authors connect Conjecture 1.1 to questions in symplectic geometry.

The main result of this paper presents a lower bound for the Mahler volume of symmetric convex bodies when the dimension of the sets satisfies $n \geq 4$. This lower bound is obtained from a variational procedure in \mathbb{R}^2 . As we observed above, the Mahler Conjecture itself can be seen as the variational problem of minimising the Mahler functional over the class of symmetric convex sets. The variational procedure with which we obtain our lower bound is rather non-trivial, but I believe it demonstrates very well the complications hidden in the Mahler Conjecture itself. Although in the present paper I do not demonstrate that this lower bound is sharp, I strongly believe it is.

Before announcing the main theorem of this paper, we'll need a few definitions. Since our lower bound is presented through a procedure in \mathbb{R}^2 , I shall also give the necessary definitions in \mathbb{R}^2 . The sets with which I work belong to a certain class of symmetric convex sets. For every $n \geq 4$

the class $\mathcal{S}(1, \sqrt{n+1})$ denotes the class of symmetric convex bodies M in \mathbb{R}^2 such that

$$B_2(0, 1) \subseteq M \subseteq B_2(0, \sqrt{n+1}).$$

I no longer will be working with the Lebesgue measure- instead I shall define a class of measures which have an anisotropic density (i.e. the density of the measures will not be radial functions- they bear weight on the circle when writing the measure in polar coordinates).

Definition 1.2 (The measures $\mu_{2,\theta}$) *The measure μ_2 is the measure defined as $r^n g(t) dr \wedge dt$ in polar coordinates of \mathbb{R}^2 . The function $g(t) = \cos(t)^{n-1}$. In Cartesian coordinates $x y$, the measure μ_2 is equal to the measure:*

$$\mu_2 = |y|^{n-1} dx dy,$$

here, the end point of the unit vector of the y -axis coincides with the maximum point of the function g . Given $\theta \in [0, \pi]$, we define the measure $\mu_{2,\theta} = r^n g(t + \theta) dr \wedge dt$. The Cartesian coordinates associated to the measure $\mu_{2,\theta}$ is the $x y$ coordinates in \mathbb{R}^2 such that the end-point of the unit vector of the y -axis coincides with the maximum point of the function $g(t + \theta)$.

We will often work on conical subsets in \mathbb{R}^2 . If I is a (geodesic) segment of \mathbb{S}_+^1 , the cone $C(I)$ over I will be the infinite cone in \mathbb{R}^2 which has the origin as its vertex and has $u_1 \mathbb{R}_+$ and $u_2 \mathbb{R}_+$ as its rays, where u_1 and u_2 are unit vectors joining the origin to the end points of I .

Definition 1.3 (Constant $\alpha(n)$) *Let $n \geq 4$. Let $I \subset \mathbb{S}^1$ be a connected interval of the circle. Let $C(I)$ be the cone in \mathbb{R}^2 over I . Let $S(I)$ be the class of all connected closed intervals of the unit circle. Let S be a 2-dimensional convex set symmetric with respect to the origin of \mathbb{R}^2 and belonging to $\mathcal{S}(1, \sqrt{n+1})$. Define*

$$\alpha(n, \theta, I, S) = \frac{\mu_{2,\theta}(C(I) \cap S) \mu_{2,\theta}(C(I) \cap S^\circ)}{(\int_I g(\theta, t) dt)^2},$$

where $\mu_{2,\theta}$ is the measure defined in 1.2 and $g(\theta, t)$ is the density of this measure in polar coordinates. Define

$$\alpha(n, S) = \max_{\theta \in [0, \pi]} \left(\min_{I \in S(I)} \alpha(n, \theta, I, S) \right).$$

And at last, define:

$$\alpha(n) = \min_S (\alpha(n, S)),$$

where S runs over the class of 2-dimensional symmetric convex sets in $\mathcal{S}(1, \sqrt{n+1})$.

For clarity, I shall explain how the constant $\alpha(n)$ is defined:

- First, fix a symmetric convex set S in \mathbb{R}^2 and in $\mathcal{S}(1, \sqrt{n+1})$ where the Cartesian coordinates are also fixed.
- Second step, fix yourself a real $\theta \in [0, \pi]$.
- According to the choosen θ , consider the measure $\mu_{2,\theta}$ now being defined on \mathbb{R}^2 .

- Find a (connected) segment $I \subset \mathbb{S}_+^1$ and consider the cone $C(I)$ over I .
- You are now prepared to calculate the following:

$$\frac{\mu_{2,\theta}(C(I) \cap S) \mu_{2,\theta}(C(I) \cap S^\circ)}{(\int_I g(\theta, t) dt)^2}.$$

This will be the value $\alpha(n, \theta, I, S)$.

- Now we are ready to follow a max-min procedure: by fixing θ , you take the min of the values $\alpha(n, \theta, I, S)$ where I will pass over connected segments on the half-circle. Then you allow θ to move in $[0, \pi]$ (which will change your measures at every step) and calculate the maximum of $\alpha(n, \theta, I, S)$. This value will be the $\alpha(n, S)$.
- All that remains now is to take $\alpha(n)$ as the minimum of the above over every symmetric convex set in $\mathcal{S}(1, \sqrt{n+1})$.

Remark:

Actually, there is an *easier* way to *calculate* $\alpha(n, S)$ by introducing a variable ϕ . Since we *rotate* the measure μ_2 , instead of letting the intervals I pass over all of the connected intervals of the half-circle, we could consider the following:

$$\alpha_1(n, S) = \max_{\theta} (\min_{\phi} \alpha(n, \theta, [0, \phi], S)),$$

where $\phi \in [0, \pi]$ and $[0, \phi]$ is an interval on the half-circle where the point 0 coincides with the point $(1, 0)$ in the Cartesian coordinates and the point ϕ coincides with the point $(\cos(\phi), \sin(\phi))$ on the half-circle. It is a trivial exercise to verify that in fact:

$$\alpha(n, S) = \alpha_1(n, S).$$

Before going any further, let me illustrate that the constant $\alpha(n)$ has a non-zero lower bound :

Lemma 1.1 *Let $\alpha(n)$ be defined as in 1.3, then we have :*

$$\alpha(n) \geq \frac{1}{(n+1)^2(n+1)^{\frac{n+1}{2}}}.$$

Proof of Lemma 1.1:

By definition, every $M \in \mathcal{S}(1, \sqrt{n+1})$ satisfies :

$$B_2(0, 1) \subseteq M \subseteq B_2(0, \sqrt{n+1}). \quad (1)$$

As a trivial exercise, one can verify that if K and L are two convex sets such that $K \subset L$, then $L^\circ \subset K^\circ$. This shows that according to (1) we have :

$$B_2(0, \frac{1}{\sqrt{n+1}}) \subseteq M^\circ.$$

Let I be any (geodesic) segment of \mathbb{S}_+^1 and let $C(I)$ be the cone over I . Let $\mu_{2,\theta}$ be a measure as in definition 1.2. Therefore we have:

$$\begin{aligned}
\alpha(n, \theta, I, K) &= \frac{\mu_{2,\theta}(C(I) \cap K) \mu_{2,\theta}(C(I) \cap K^\circ)}{(\int_I g(\theta, t) dt)^2} \\
&\geq \frac{\mu_{2,\theta}(C(I) \cap B(0, 1)) \mu_{2,\theta}(C(I) \cap B(0, 1/\sqrt{n+1}))}{(\int_I g(\theta, t) dt)^2} \\
&= \frac{(\int_I g(\theta, t) dt)^2}{(n+1)^2 (n+1)^{\frac{n+1}{2}} (\int_I g(\theta, t) dt)^2} \\
&= \frac{1}{(n+1)^2 (n+1)^{\frac{n+1}{2}}}.
\end{aligned}$$

Since the lower bound obtained for $\alpha(n, \theta, I, K)$ is independent of θ , I and K , we have :

$$\alpha(n) \geq \frac{1}{(n+1)^2 (n+1)^{\frac{n+1}{2}}}.$$

This proves Lemma 1.1. □

After that lengthy introduction, we are now ready to introduce the main theorem of this paper:

Theorem 2 *Let K be a symmetric convex set in \mathbb{R}^n where $n \geq 4$. Then*

$$\begin{aligned}
\text{vol}(K) \text{vol}(K^\circ) &\geq \alpha(n-1) \text{vol}_{n-1}(\mathbb{S}^{n-1})^2 \\
&= \frac{4\alpha(n-1)\pi^n}{\Gamma(\frac{n}{2})^2},
\end{aligned}$$

where $\alpha(n)$ is the constant defined in 1.3.

Remark:

Applying Lemma 1.1 and Theorem 2, we obtain the following lower bound for the Mahler volume of every (at least four dimensional) symmetric convex set K in \mathbb{R}^n :

$$\text{vol}(K) \text{vol}(K^\circ) \geq \frac{4\pi^n}{n^{\frac{n+4}{2}} \Gamma(\frac{n}{2})^2}.$$

From this theorem, two different paths open up:

- First we attempt to calculate $\alpha(n)$ for $n \geq 4$ (for which I am less enthusiastic).
- Second, we attempt to prove that there exists a symmetric convex set $n \geq 4$ for which its Mahler volume is *exactly* equal to the lower bound given by Theorem 2 (in which I believe strongly).

It is then highly possible that :

Conjecture 1.2 *Let $n \geq 4$. There is a symmetric convex set K in \mathbb{R}^n for which we have*

$$\text{vol}_n(K)\text{vol}_n(K^\circ) = \frac{4\alpha(n-1)\pi^n}{\Gamma(\frac{n}{2})^2}.$$

Remark:

Another quick observation is that when the cone $C(I)$ is reduced to a point, the value of $\alpha(n, \theta, I, S)$ for every S and every θ is equal to 1 (i.e. set $\frac{0}{0} = 1$) and therefore:

$$\begin{aligned} \frac{4\alpha(n, I, S)\pi^n}{\Gamma(\frac{n}{2})^2} &= \frac{4\pi^n}{\Gamma(\frac{n}{2})^2} \\ &> \frac{4^n}{\Gamma(n+1)}. \end{aligned}$$

The second case to consider would be when $C(I)$ is the half-plane. Even in this case I do not know a sharp lower bound, however in the last section I will discuss this case and present a conjecture for an interesting lower bound for $\alpha(n, \theta, I, S)$, where I is the half-circle \mathbb{S}_+^1 . The rest of this paper is devoted to the proof of Theorem 2. To achieve this goal, we shall first require several techniques (which we shall learn in Section 3). There, I shall recall the theory of convexly-derived measures, measurable partitions and the localisation on the canonical Riemannian sphere. Section 4 will deal with the proof of Theorem 2. And finally, Section 5 contains further remarks concerning Theorem 2 and Conjecture 1.2.

2 Acknowledgement

Not long ago, I was busy working on the Gaussian Correlation Conjecture (in [38]). I had the idea of using spherical localisation to simplify the Conjecture to a 2-dimensional problem. During that period, I had a dream one night in which an unknown person presented me with the Mahler Conjecture and told me that the spherical localisation may also be used to prove this conjecture. The first thing I did when I woke up was to Google *Mahler Conjecture*. Before this dream, I had no idea what this conjecture was about, and I don't recall any mathematician talking to me about this conjecture *before* the dream (perhaps I had read it somewhere and not paid attention to it?). The idea tickled me immensely and kept my brain occupied about whether or not spherical localisation could help in proving this conjecture. Three years later, I finally realised that the man in my dream was (at least partially) right; spherical localisation *indeed* helps to obtain a result for this conjecture. Apart from the man in my dream, I have to also thank F.Barthe, M.Fradellizi and R.Vershynin for their helpful emails answering a few of my questions. At last, I truly thank my wife for proof-reading my paper (as she does with all my papers!), for correcting my awful English, and for making this paper readable and presentable.

3 Localisation on the Sphere

In the past years, localisation methods have been used to prove several very interesting geometric inequalities. In [30] and [24], the authors proved integral formulae using localisation, and applied their methods to conclude a few isoperimetric-type inequalities concerning convex sets in the Euclidean space. In [12] the authors study a functional analysis version of the localisation, used again on the Euclidean space. The authors of [12] presented to me a proof of the (well known)

Prekopa-Leindler Inequality using their localisation result in [12]. Localisation on more general spaces was also studied in [19], [18], [36], and [37].

I shall begin with some definitions and remind the reader that much of the material in this section is derived from [36]:

Definition 3.1 (Convexly-derived measures) *A convexly-derived measure on \mathbb{S}^n (resp. \mathbb{R}^n) is a limit of a vaguely converging sequence of probability measures of the form $\mu_i = \frac{\text{vol}(S_i)}{\text{vol}(\mathbb{S}^n)}$, where S_i are open convex sets. The space \mathcal{MC}^n is defined to be the set of probability measures on \mathbb{S}^n which are of the form $\mu_S = \frac{\text{vol}(S)}{\text{vol}(\mathbb{S}^n)}$ where $S \subset \mathbb{S}^n$ is open and convex. The space of convexly-derived probability measures on \mathbb{S}^n is the closure of \mathcal{MC}^n with respect to the vague (or weak by compactness of \mathbb{S}^n)-topology. The space \mathcal{MC}^k will be the space of convexly-derived probability measures whose support has dimension k and $\mathcal{MC}^{\leq k} = \cup_{l=0}^k \mathcal{MC}^l$.*

This class of measures was defined first in [19] and used later on in [1], [36], [35]. In Euclidean spaces, a convexly-derived measure is simply a probability measure supported on a convex set which has a x^k -concave density function with respect to the Lebesgue measure. To understand convexly-derived measures on the sphere we will need some definitions:

Definition 3.2 (sin-concave functions) *A real function f (defined on an interval of length less than 2π) is called sin-concave, if, when transported by a unit speed parametrisation of the unit circle, it can be extended to a 1-homogeneous and concave function on a convex cone of \mathbb{R}^2 .*

Definition 3.3 (\sin^k -affine functions and measures) *A function f is affinely \sin^k -concave if $f(x) = A \sin^k(x + x_0)$ for a $A > 0$ and $0 \leq x_0 \leq \pi/2$. A \sin^k -affine measure by definition is a measure with a \sin^k -affine density function.*

Definition 3.4 (\sin^k -concave functions) *A non-negative real function f is called \sin^k -concave if the function $f^{\frac{1}{k}}$ is sin-concave.*

One can easily confirm the following:

Lemma 3.1 *A real non-negative function defined on an interval of length less than π is \sin^k -concave if for every $0 < \alpha < 1$ and for all $x_1, x_2 \in I$ we have*

$$f^{1/k}(\alpha x_1 + (1 - \alpha)x_2) \geq \left(\frac{\sin(\alpha|x_2 - x_1|)}{\sin(|x_2 - x_1|)} \right) f(x_1)^{1/k} + \left(\frac{\sin((1 - \alpha)|x_2 - x_1|)}{\sin(|x_2 - x_1|)} \right) f(x_2)^{1/k}.$$

Particularly if $\alpha = \frac{1}{2}$ we have

$$f^{1/k}\left(\frac{x_1 + x_2}{2}\right) \geq \frac{f^{1/k}(x_1) + f^{1/k}(x_2)}{2 \cos\left(\frac{|x_2 - x_1|}{2}\right)}.$$

One can use Lemma 3.1 as a definition of \sin^k -concave functions. This class of measures are also used in Optimal Transport Theory (see the excellent book [57] on this matter as well as a proof for Lemma 3.1).

Lemma 3.2 *Let S be a geodesically convex set of dimension k of the sphere \mathbb{S}^n with $k \leq n$. Let μ be a convexly-derived measure defined on S (with respect to the normalised Riemannian measure on the sphere). Then μ is a probability measure having a continuous density f with respect to the canonical Riemannian measure on \mathbb{S}^k restricted to S . Furthermore, the function f is \sin^{n-k} -concave on every geodesic arc contained in S .*

The above lemma (proved in [36]) completely characterises the class of convexly-derived measures on the sphere. Note the similarity between the Euclidean case and the spherical one.

Before we begin a study of the properties of \sin^k -concave functions, I shall define the definition of *spherical needles*.

Definition 3.5 (Spherical Needles) *A spherical needle in \mathbb{S}^n is a couple (I, ν) where I is a geodesic segment in \mathbb{S}^n and ν is a probability measure supported on I which has a \sin^{n-1} -affine density function.*

Remark:

Form definition 3.5, we can properly write down the measure ν . To do so, choose a parametrisation of the geodesic segment I by its arc length. Therefore there is a (canonical) map $s : [0, l(I)] \rightarrow I$. For every $t \in [0, l(I)]$, we have that $\|\frac{ds}{dt}\| = 1$. The measure dt is the canonical Riemannian length-measure associated to the geodesic segment I . Using now the parametrisation of I by $t \in [0, l(I)]$, the measure ν can be written as $\nu = C \cos^{n-1}(t - t_0)dt$, for $t_0 \in [0, \pi]$ and C the normalisation constant such that :

$$\int_0^{l(I)} C \cos(t - t_0)^{n-1} dt = 1.$$

3.1 Some Useful Properties of \sin^k -Concave Measures/Functions

The next lemma quickly reviews some properties of \sin^k -concave functions. Mainly, the results suggest that in many cases, \sin^k -concave functions are *nicey comparable* to \sin^k -affine functions. Though we do not really need the result of the next lemma for the rest of the paper, I think it helps to give a glimpse to the reader of how these functions behave.

Lemma 3.3 • *f admits only one maximum point and does not have any local minima.*

- *If f is \sin -concave and defined on an interval containing 0, then $g(t) = f(|t|)$ is also \sin -concave.*
- *Let $0 < \varepsilon < \pi/2$. Let $\tau > \varepsilon$. f is defined on $[0, \tau]$ and attains its maximum at 0. Let $h(t) = c \cos(t)^k$ where c is choosen such that $f(\varepsilon) = h(\varepsilon)$. Then*

$$\begin{cases} f(x) \geq h(x) & \text{for } x \in [0, \varepsilon], \\ f(x) \leq h(x) & \text{for } x \in [\varepsilon, \tau]. \end{cases}$$

In particular, $\tau \leq \pi/2$.

- *Let $\tau > 0$ and f be a nonzero non-negative \sin^k -concave function on $[0, \tau]$ which attains its maximum at 0. Then $\tau \leq \pi/2$ and for all $\alpha \geq 0$ and $\varepsilon \leq \pi/2$ we have*

$$\frac{\int_0^{\min\{\varepsilon, \tau\}} f(t) dt}{\int_0^\tau f(t) dt} \geq \frac{\int_0^\varepsilon \cos(t)^k dt}{\int_0^{\pi/2} \cos(t)^k dt}.$$

3.2 A Fundamental Spherical Localisation Lemma

The main result of this section shall be Lemma 3.6. I prefer to arrive at this lemma progressively. First, I would like to prove a simplified version of this lemma (which shall be Lemma 3.4) and which had its Euclidean counterpart proved first in [30]:

Lemma 3.4 *Let G_i for $i = 1, 2$ be two continuous functions on \mathbb{S}^n such that*

$$\int_{\mathbb{S}^n} G_i(u) d\mu(u) > 0,$$

then a \sin^{n-1} -affine probability measure ν supported on a geodesic segment I exists such that

$$\int_I G_i(t) d\nu(t) > 0.$$

Proof of Lemma 3.4

We construct a decreasing sequence of convex subsets of \mathbb{S}^n using the following procedure:

- Define the first step cutting map $F_1 : \mathbb{S}^n \rightarrow \mathbb{R}^2$ by

$$F_1(x) = \left(\int_{x^\vee} G_1(u) d\mu(u), \int_{x^\vee} G_2(u) d\mu(u) \right)$$

where x^\vee denotes the (oriented) open hemi-sphere pointed by the vector x . Apply Borsuk-Ulam Theorem to F_1 . Hence there exists a x_1^\vee such that

$$\begin{aligned} \int_{x_1^\vee} G_1(u) d\mu(u) &= \int_{-x_1^\vee} G_1(u) d\mu(u) \\ \int_{x_1^\vee} G_2(u) d\mu(u) &= \int_{-x_1^\vee} G_2(u) d\mu(u). \end{aligned}$$

Choose the hemi-sphere, denoted by x_1^\vee . Set $S_1 = x_1^\vee \cap \mathbb{S}^n$.

- Define the i -th step cutting map by

$$F_i(x) = \left(\int_{S_{i-1} \cap x^\vee} G_1(u) d\mu(u), \int_{S_{i-1} \cap x^\vee} G_2(u) d\mu(u) \right).$$

By applying the Borsuk-Ulam Theorem to F_i , we obtain two new hemi-spheres and we choose the one, denoted by x_i^\vee . Set $S_i = x_i^\vee \cap S_{i-1}$.

This procedure defines a decreasing sequence of convex subsets $S_i = x_i^\vee \cap S_{i-1}$ for every $i \in \mathbb{N}$. Set:

$$S_\pi = \bigcap_{i=1}^{\infty} (S_i) = \bigcap_{i=1}^{\infty} \text{clos}(S_i),$$

where $\text{clos}(A)$ determines the topological closure of the subset A . We call the hemi-spheres obtained from the cutting maps, *cutting hemi-spheres*. Furthermore, a convexly-derived probability measure

ν_π is defined on S_π . Since $\lim_{i \rightarrow \infty} S_i = S_\pi$ (this limit is with respect to Hausdorff topology) the definition of the convexly-derived measures can be applied to define the positive probability measure supported on S_π by

$$\nu_\pi = \lim_{i \rightarrow \infty} \frac{\mu|_{S_i}}{\mu(S_i)}.$$

Hence, by the definition of ν_π

$$\int_{S_\pi} G_j(x) d\nu_\pi(x) = \lim_{i \rightarrow \infty} \frac{\int_{S_i} G_j(x) d\mu(x)}{\mu(S_i)}$$

for $j = 1, 2$, and where the limit is taken with respect to the vague topology defined on the space of convexly-derived measures (see [36]). Recall the following :

Lemma 3.5 (See [21]). *Let μ_i be a sequence of positive Radon measures on a locally-compact space X which vaguely converges to a positive Radon measure μ . Then, for every relatively compact subset $A \subset X$, such that $\mu(\partial A) = 0$,*

$$\lim_{i \rightarrow \infty} \mu_i(A) = \mu(A).$$

By the definition of the cutting maps $F_i(x)$, for every $i \in \mathbb{N}$, $j = 1, 2$ we have

$$\int_{S_i} G_j(u) d\mu(u) > 0.$$

By applying Lemma 3.5, we conclude that the convexly-derived probability measure defined on S_π satisfies the assumption of the Lemma 3.4. The dimension of S_π is $< n$. Indeed, if it is not the case, then $\dim S_\pi = n$. Since there is a convexly-derived measure with positive density defined on S_π , and by the construction of the sequence $\{S_i\}$ for every open set U we have

$$\begin{aligned} \nu_\pi(S_\pi \cap U) &= \lim_{i \rightarrow \infty} \frac{\mu(S_i \cap U)}{\mu(S_i)} \\ &= \lim_{i \rightarrow \infty} \frac{\mu(S_\pi \cap U)}{2^i \mu(S_i)}. \end{aligned}$$

By supposition on the dimension of S_π , the right-hand equality is equal to zero. This is a contradiction with the positive measure ν_π charging mass on $S_\pi \cap U$.

Thus, $\dim S_\pi < n$. If $\dim S_\pi = 1$ then Lemma 3.4 is proved. Note that $\dim S_\pi$ can not be equal to zero, since the cutting map in each step cuts the set S_i . If $\dim S_\pi = k > 2$, we define a new procedure by replacing S with $S_\pi \cap S$, replacing the normalized Riemannian measure by the measure ν_π , and replacing the sphere \mathbb{S}^n by the sphere \mathbb{S}^k containing S_π . For this new procedure, we define new cutting maps in each step. Since $k > 2$, by using the Borsuk-Ulam Theorem we obtain hyperspheres (\mathbb{S}^{k-1}) halving the desired (convexly-derived) measures. The new procedure defines a new sequence of convex subsets and, by the same arguments given before, a convexly-derived measure defined on the intersection of this new sequence satisfying the assumption of Lemma 3.4. By the same argument, the dimension of the intersection of the decreasing sequence of convex sets is $< k$. If the dimension of the intersection of this new sequence is equal to 1, we are finished. If not, we repeat the above procedure until arriving to a 1-dimensional set. This proves

that a probability measure ν with a (non-negative) \sin^{n-1} -concave density function f , supported on a geodesic segment I exists such that:

$$\int_I f(t)G_i(t)dt \geq 0. \quad (2)$$

We determine I to have minimal length. If f is \sin^{n-1} -affine on I then we are done. We suppose this is not the case. We choose a subinterval $J \subset I$, maximal in length, such that a \sin^{n-1} -concave function f satisfying (2) exists such that f additionally is \sin^{n-1} -affine on the subinterval J . The existence of J and f follows from a standard compactness argument. We can assume that the length of I is $< \pi/2$. Consider the Euclidean cone over I . Let $a, b \in I$ be the end points of I and take the *Euclidean segment* $[a, b]$ in \mathbb{R}^2 (basically the straight line joining a to b). By definition of \sin^{n-1} -concave functions, the function f is the restriction of a one-homogeneous x^{n-1} -concave function F on the circle (a x^{n-1} -concave function F is a function such that $F^{1/(n-1)}$ is concave). Transporting the entire problem to \mathbb{R}^{n+1} , we begin with two homogeneous functions \bar{G}_i on \mathbb{R}^{n+1} such that

$$\int_{\mathbb{R}^{n+1}} \bar{G}_i dx > 0$$

and we proved that there is a 2-dimensional cone over a segment $[a, b]$, a one-homogeneous x^{n-1} -concave function F on $[a, b]$, and a subinterval $[\alpha, \beta] \subset [a, b]$ such that $F^{1/(n-1)}$ is linear on $[\alpha, \beta]$ (this is due to the fact that by definition, the restriction of a one-homogeneous x^{n-1} -affine function on a 2-dimensional Euclidean cone defines a \sin^{n-1} -affine function on the circle) and such that

$$\int_{[\alpha, \beta]} \bar{G}_i(t)F(t)dt \geq 0.$$

We echo the arguments given in [30] (pages 21 – 23) (with the only difference being that every construction there drops by one dimension). This drop of dimension is necessary so that that every construction may preserve homogeneity- or in other words- one dimension must be preserved for the 2-dimensional cone defined on $[a, b]$. Hence the proof of Lemma 3.4 follows.

□

Though Lemma 3.4 provides us (literally) with *one* spherical needle preserving the positivity of both integrals, we require much more than that. And, it appears that in fact there are *many* of these spherical needles- each preserving the positivity of both the integrals. Even better, these spherical needles rise together to give a (measurable) partition of the canonical sphere. It is then necessary to say a few words on *convex partitions* of the canonical sphere \mathbb{S}^n before going further.

Definition 3.6 Let Π be a finite convex partition of \mathbb{S}^n . We review this partition as an atomic probability measure $m(\Pi)$ on the space \mathcal{MC} as follows: for each piece S of Π , let $\mu_S = \frac{\text{vol}_S}{\text{vol}(\mathbb{S}^n)}$ be the normalised volume of S . Then set

$$m(\Pi) = \sum_S \frac{\text{vol}(S)}{\text{vol}(\mathbb{S}^n)} \delta_{\mu_S}.$$

Define the space of (infinite) convex partitions \mathcal{CP} as the vague closure of the image of the map m in the space $\mathcal{P}(\mathcal{MC})$ of probability measures on the space of convexly-derived measures. The subset $\mathcal{CP}^{\leq k}$ of convex partitions of dimension $\leq k$ consists of elements of \mathcal{CP} which are supported on the subset $\mathcal{MC}^{\leq k}$ of convexly derived measures with support of dimension (at most) k . It is worth remembering that the space \mathcal{CP} is compact and $\mathcal{CP}^{\leq k}$ is closed within.

We are now prepared to state the next lemma which will be the generalisation of Lemma 3.4 and is the main result of this section :

Lemma 3.6 *Let G_i for $i = 1, 2$ be two continuous functions on \mathbb{S}^n such that*

$$\int_{\mathbb{S}^n} G_i(u) d\mu(u) > 0,$$

then a convex partition of \mathbb{S}^n , $\Pi \in \mathcal{CP}^{\leq 1}$ by geodesic segments exists such that for every σ an element of Π , we have

$$\int_{\sigma} G_i(t) d\nu_{\sigma}(t) > 0.$$

ν_{σ} is a \sin^{n-1} -concave probability measure which is canonically defined on σ from the partition.

Proof of Lemma 3.6 :

The proof of this lemma is similar to the proof of Lemma 3.4. In the proof of Lemma 3.4, we used a family of $\{x_i^{\vee}\}$ of oriented hemi-spheres to cut the sphere. Each x_i^{\vee} cuts the sphere in two parts in such a way that in *both* parts of the sphere and the positivity of the integral of G_i is respected (due to the Borsuk-Ulam Theorem). At each stage of cutting, we only keep one part of the sphere. However, if we carry out everything we did with respect to *other* parts, we obtain (in a straightforward way) the conclusion of Lemma 3.6. We should keep in mind that the partition $\Pi \in \mathcal{CP}^{\leq 1}$ can be constructed by choosing the family of cutting hemi-spheres $\{x_i^{\vee}\}$ such that all the vectors x_i belong to a sphere of dimension k provided $k \geq 2$. This has two benefits:

- The partition obtained in Lemma 3.6 is *not* unique.
- We can choose the *direction* of the cuts by appropriately choosing the sphere \mathbb{S}^k . This fact will become very useful in the proof of Theorem 2.

□

Remark:

Instead of applying the Borsuk-Ulam Theorem in the proof of Lemma 3.6 we can use the more powerful Gromov-Borsuk-Ulam Theorem which is stated and proved in [36]. The Gromov-Borsuk-Ulam Theorem provides a convex partition $\mathcal{CP}^{\leq k}$ for every continuous map $f : \mathbb{S}^n \rightarrow \mathbb{R}^k$ where $k < n$ and a point $z \in \mathbb{R}^k$ such that $f^{-1}(z)$ intersects the maximum points of the density of the convexely-derived-measures associated to the partition. A drop in the dimension k is possible if we do not assume that for every cut, the volume of the convex sets has to be preserved. Therefore by applying the Gromov-Borsuk-Ulam Theorem directly for the map $f : \mathbb{S}^n \rightarrow \mathbb{R}^2$ defined by

$$f(x) = \left(\int_{x^{\vee}} G_1(u) d\mu(u), \int_{x^{\vee}} G_2(u) d\mu(u) \right),$$

we obtain the desired convex partition $\Pi \in \mathcal{CP}^{\leq 1}$ of the Lemma 3.6.

It is interesting to compare the partition result of Lemma 3.6 with the Yao-Yao partition (see [59] and [28] for this matter).

We'll now include few useful definitions and remarks which we shall need for the proof of Theorem 2.

Definition 3.7 (Pancakes) Let S be an open convex subset of \mathbb{S}^n . Let $\varepsilon > 0$. We call S an (k, ε) -pancake if there exists a convex set S_π of dimension k such that every point of S is at distance at most ε from S_π .

Remark:

Every geodesic segment I (an element of a partition obtained from Lemma 3.6) is a Hausdorff limit of a sequence $\{S_i\}_{i=1}^\infty$, where S_i is a $(1, \varepsilon_i)$ -pancake and where $\varepsilon_i \rightarrow 0$ when $i \rightarrow \infty$. Furthermore, every spherical needle (I, ν) (also an element of the partition obtained in Lemma 3.6) is a limit of a sequence of $(1, \varepsilon_i)$ -pancakes, where the measure ν is a (weak)-limit of the sequence of probability measures obtained by normalising the volume of each pancake.

Definition 3.8 (Constructing Pancake) Let (I, ν) be a spherical needle. We call a $(1, \delta)$ -pancake S , a constructing pancake for (I, ν) , if there exists a decreasing sequence of pancakes $\dots \subset S_i \subset S_{i-1} \subset \dots \subset S_0$, where $S_0 = S$ and (I, ν) is a limit of this sequence.

Definition 3.9 (Distance Between Spherical Needles) For $\varepsilon > 0$, we say that the distance between the spherical needles (I_1, ν_1) and (I_2, ν_2) is at most equal to ε if there exists a constructing pancake S_1 (resp S_2) for (I_1, ν_1) (resp (I_2, ν_2)) such that S_1 (resp S_2) is in the ε -neighborhood of I_1 (resp I_2) and the Hausdorff distance between S_1 and S_2 is at most equal to ε .

Definition 3.10 (Cutting Hemi-spheres) A cutting hemi-sphere is a \mathbb{S}_+^n , which is a hemi-sphere used at some stage of the algorithmic procedure of obtaining a partition given by Lemma 3.6. The first cutting hemi-sphere will be the hemi-sphere used at the very first stage of the procedure to cut the sphere \mathbb{S}^n into two parts.

4 Proof of Theorem 2

The Mahler volume is invariant under linear mappings. According to John's Lemma (see [23] or [56]), for every n -dimensional symmetric convex set M , there exists an ellipsoid of maximal volume E contained in M , and such that M is contained in $\sqrt{n}E$. Therefore, in order to calculate the Mahler volume of M , it is enough to calculate the Mahler volume of $L(M)$, where L is an appropriate linear map and such that the ellipsoid of maximal volume contained in $L(M)$ coincides with $B_n(0, 1)$. Hence, we will not lose any information by only evaluating the Mahler volume of the class of symmetric convex bodies in \mathbb{R}^n for which their John's Ellipsoid is the unit ball $B_n(0, 1)$. With this remark, we can then suppose M is a symmetric convex body in \mathbb{R}^n and

$$B_n(0, 1) \subseteq M \subseteq B_n(0, \sqrt{n}).$$

To prove Theorem 2, we shall proceed by contradiction. For simplicity we denote:

$$\beta(n) = \alpha(n-1) \text{vol}_{n-1}(\mathbb{S}^{n-1})^2.$$

Assume we have a symmetric convex set in \mathbb{R}^n such that:

$$\text{vol}_n(K) \text{vol}_n(K^\circ) < \beta(n). \quad (3)$$

Consider the normalised Riemannian volume on \mathbb{S}^{n-1} which shall be denoted by $d\mu$. Therefore we have:

$$\frac{\text{vol}_n(K)}{\text{vol}_{n-1}(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} F_1(u) d\mu(u), \quad (4)$$

where $F_1(u) = \frac{x(u)^n}{n}$ and $x(u)$ is the length of the segment issuing from the origin in the direction of u and touching the boundary of K . Similarly, we have:

$$\frac{\text{vol}_n(K^\circ)}{\text{vol}_{n-1}(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} F_2(u) d\mu(u), \quad (5)$$

where the function F_2 is defined similar to the function F_1 .

Combining equations (3), (4) and (5), we obtain :

$$\left(\int_{\mathbb{S}^{n-1}} F_1(u) d\mu(u) \right) \left(\int_{\mathbb{S}^{n-1}} F_2(u) d\mu(u) \right) < \alpha(n-1). \quad (6)$$

Thus, there exists a $C > 0$ such that:

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} F_1(u) d\mu(u) &< C \\ &< \frac{\alpha(n-1)}{\int_{\mathbb{S}^{n-1}} F_2(u) d\mu(u)}. \end{aligned}$$

Define two functions G_1 and G_2 on \mathbb{S}^{n-1} as follows. For every $u \in \mathbb{S}^{n-1}$,

$$G_1(u) = C - F_1(u),$$

and

$$G_2(u) = \alpha(n-1) - C F_2(u).$$

Therefore, we have defined two continuous functions G_1 and G_2 on \mathbb{S}^{n-1} such that (for every $i = 1, 2$) we have:

$$\int_{\mathbb{S}^{n-1}} G_i(u) d\mu(u) > 0.$$

We apply Lemma 3.6 which demonstrates the existence of a partition of \mathbb{S}^{n-1} into spherical needles, such that for every spherical needle (I, ν) we have:

$$\int_I G_i(t) d\nu(t) > 0.$$

This translates to the fact that for every such spherical needle (I, ν) , we have :

$$\left(\int_I F_1(t) d\nu(t) \right) \left(\int_I F_2(t) d\nu(t) \right) < \alpha(n-1).$$

Take a 2-dimensional cone over I . Writing the above integral inequality on this cone (and appropriately choosing the x, y -coordinates), would give us:

$$\left(\int_{C(I) \cap K} |x|^{n-1} dx dy \right) \left(\int_{C(I) \cap K^\circ} |x|^{n-1} dx dy \right) < \alpha(n-1).$$

We need to remember that even if we translated the integrals on spherical needles we do not always have the following:

$$C(I) \cap K^\circ \neq (C(I) \cap K)^\circ.$$

A well known fact is that if K is a symmetric convex set in \mathbb{R}^n and if \mathbb{R}^k is a k -dimensional vector subspace of \mathbb{R}^n , then:

$$(\mathbb{R}^k \cap K)^\circ = P_{\mathbb{R}^k}(K^\circ),$$

where $P_{\mathbb{R}^k}(K^\circ)$ is the orthogonal projection of K° in \mathbb{R}^k (one can prove this fact as an exercise or consult the proof in [56]). We all know that projecting K° in a k -dimensional vector subspace provides us with a convex set which can be very different from taking the intersection $K^\circ \cap \mathbb{R}^k$. However, we can always find a 2-dimensional vector-subspace of \mathbb{R}^n for which the projection and intersection coincides. One way to obtain such an intersection is as follows: Consider a convex body with a smooth boundary inscribed in K° (one can take the John Ellipsoid for example). Consider the tangent bundle of the boundary of this convex body and take a 2-dimensional vector sub-space of \mathbb{R}^n , which cuts the tangent bundle orthogonally. This particular 2-dimensional vector sub-space can be use as the desired section for which the projection and section of K° onto it coincides with each other. Hence we have a \mathbb{R}^2 such that :

$$P_{\mathbb{R}^2}K^\circ = K^\circ \cap \mathbb{R}^2. \quad (7)$$

Let $L = \mathbb{R}^2$ be the 2-dimensional vector sub-space in (7). Denote by $\sigma = \mathbb{S}^1 = L \cap \mathbb{S}^{n-1}$. For every $\theta \in \sigma$, we define a (canonical) maximal spherical needle (I_θ, ν_θ) . The geodesic segment I_θ is the half-circle $[\theta - \pi/2, \theta + \pi/2]$. The measure ν_θ is the \sin^{n-2} -affine probability measure such that the maximum of the density function of this measure coincides with the point θ . We see every segment J (which is a subset of I_θ) as a spherical needle $(J, \nu_\theta|J)$, where $\nu_\theta|J$ is the restriction of the measure ν_θ on J , then we normalise it on J . Be aware (again) that the same geodesic segment can be seen as different spherical needles. Let θ_0 be the one maximising $\alpha((n-1), \theta, J, L \cap K)$ (remember the definition from the introduction). Consider the (maximal) spherical needle $(I_{\theta_0}, \nu_{\theta_0})$. For simplicity of notation from now on, I shall denote this spherical needle by (I_0, ν_0) and the center of this spherical needle (i.e. the point θ_0) by x_0 .

Let U be the hemi-sphere (i.e. \mathbb{S}_+^{n-1}) centered at the point x_0 . Therefore $I_0 \subset U$. Since we are dealing with symmetric convex bodies, for every hemi-sphere U of \mathbb{S}^{n-1} containing I_0 and for $i = 1, 2$ we have :

$$\begin{aligned} \int_U G_i(u) d\mu(u) &= \frac{1}{2} \int_{\mathbb{S}^{n-1}} G_i(u) d\mu(u) \\ &> 0. \end{aligned}$$

Therefore, we can consider U as the first *cutting hemi-sphere* (see definition 3.10) of \mathbb{S}^{n-1} and then apply Lemma 3.6 to obtain a partition of U satisfying the desired assumptions. Let us denote such a partition by $\Pi(U)$. (Be aware that there are plenty of $\Pi(U)$).

Now comes the second difficulty (what was the first?): We can not guarantee that we shall have a spherical needle in $\Pi(U)$ which coincides with a spherical needle $(J, \nu_0|J)$ for a certain $J \subset I_0$. However, keeping the notations similar as above, we have the following :

Lemma 4.1 *For every $\varepsilon > 0$, there exists a partition of $\Pi(U)$ such that there exists a spherical needle (I, ν) in $\Pi(U)$ which is ε -close to a spherical needle $(J, \nu_0|J)$.*

Proof of Lemma 4.1 :

We proceed by contradiction. Assume there is an $\varepsilon_0 > 0$ such that for every spherical needle $(J, \nu_0|J)$ where $J \subset I_0$ and every spherical needle (I, ν) in a partition $\Pi(U)$ (given by Lemma 3.6) we have :

$$d((J, \nu_0|J), (I, \nu)) > \varepsilon_0. \quad (8)$$

Take a (J, ν_J) such that $J \subset I_0$ and (I, ν) in a $\Pi(U)$ such that :

$$d((J, \nu_0|J), (I, \nu)) = \varepsilon,$$

and assume ε is the minimal distance obtained by the pair $((J, \nu_0|J), (I, \nu))$.

Let $(I_1, \nu_1) = (J, \nu_0|J)$ and $(I_2, \nu_2) = (I, \nu)$. Let $\delta > 0$ be such that $\delta < \varepsilon_0/2$. We choose K_1 (resp K_2) a $(1, \delta)$ -constructing pancake for (I_1, ν_1) (resp (I_2, ν_2)) such that $I_1 \subset K_1 \subset I_1 + \delta$ and $I_2 \subset K_2 \subset I_2 + \delta$. Therefore by definition, the Hausdorff distance between K_1 and K_2 is larger than ε_0 . At the same time, by the minimality of ε , the intersection of K_1 and K_2 is non-empty. We now start to cut the K_2 pancake in an appropriate manner. Choose the cutting hemi-sphere of K_2 to be a hemi-sphere with its boundary being a $(n - 2)$ -dimensional sphere which is orthogonal to the geodesic segment τ_2 (which is the longest geodesic inside K_1 that is orthogonal to I_1). We start to cut K_2 with respect to this cutting hemi-sphere. After the first cut, we choose a new set K_2^1 , which is the intersection of K_2 with the cutting hemi-sphere containing the geodesic I_1 . We carry on the cutting process (each time with respect to a cutting-hemi-sphere such that its boundary is orthogonal to τ_2) and at each stage we choose the hemi-sphere containing the geodesic I_1 . We denote the set obtained after j times cutting by K_2^j . By the choice of the cutting hemi-spheres, the distance between the geodesic I_1 to the set K_2^j is non-increasing as j increases. Hence, eventually, there will be a K_2^j pancake such that its distance to a subset of the geodesic I_1 would be less than δ . Denote this subset of I_1 by J_1 and consider it as a spherical needle where the probability measure on J_1 is the probability measure obtained by restricting the measure ν_1 on J_1 and normalising it. Moreover, this pancake will be a subset of $(I_1 \cap K_1 \cap K_2) + \delta$, and hence a spherical needle inside K_2^j which will be δ -close to $(J_1, \nu_1|_{J_1})$, where $J_1 \subset I_1$. This is a contradiction with the definition of ε_0 . The proof of Lemma 4.1 is completed. □

A cut of \mathbb{S}^{n-1} is a *Borsuk-Ulam* solution of a map from a \mathbb{S}^k to \mathbb{R}^2 (see the proof of Lemma 3.4). To have such a solution, we need to have at least $k \geq 2$. During the proof of Lemma 4.1, we had an additional *geodesic* (which was obtained from the intersection of the 2-dimensional space L satisfying (7) with \mathbb{S}^{n-1}). We obtained the cuts in the proof of Lemma 4.1 by taking spheres orthogonal to this *geodesic*. Therefore our space \mathbb{S}^{n-1} can be at least of dimension 3 and therefore $n \geq 4$. This is the restriction on the dimension given in our Theorem 2.

Define a function

$$F : \mathcal{MC}^{\leq 1} \rightarrow \mathbb{R}_+,$$

where $\mathcal{MC}^{\leq 1}$ is the space of *spherical needles* by

$$F((I, \nu)) = \left(\int_I F_1(t) d\nu(t) \right) \left(\int_I F_2(t) d\nu(t) \right).$$

Since the space $\mathcal{MC}^{\leq 1}$ is compact, and since the function F is a continuous function on $\mathcal{MC}^{\leq 1}$, for every $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that for every pair of spherical needles (I_1, ν_1) and (I_2, ν_2) which are ε -close to each other, we have :

$$\left(\int_{I_1} F_1(t) d\nu_1(t) \right) \left(\int_{I_1} F_2(t) d\nu_1(t) \right) \geq C(\varepsilon) \left(\int_{I_2} F_1(t) d\nu_2(t) \right) \left(\int_{I_2} F_2(t) d\nu_2(t) \right).$$

According to Lemma 4.1, for every $\varepsilon > 0$, there exists a $\Pi(U)$ and a spherical needle (I, ν) in $\Pi(U)$ and a spherical needle $(J, \nu_0|J)$ which are ε -close to each other. Therefore we obtain :

$$\begin{aligned} \left(\int_I F_1(t) d\nu(t) \right) \left(\int_I F_2(t) d\nu(t) \right) &\geq C(\varepsilon) \left(\int_J F_1(t) d\nu_J(t) \right) \left(\int_J F_2(t) d\nu_J(t) \right) \\ &\geq C(\varepsilon) \alpha(n-1). \end{aligned}$$

The second inequality follows from the definition of $\alpha(n-1)$ which is the minimum of the values the product of the integrals of F_1 and F_2 can take.

We clearly have $\lim_{\varepsilon \rightarrow 0} C(\varepsilon) = 1$ and therefore for ε small enough, and following the result of Lemma 4.1, we have spherical needle (I, ν) satisfying the assumption of Lemma 3.6 such that :

$$\left(\int_I F_1(t) d\nu(t) \right) \left(\int_I F_2(t) d\nu(t) \right) \geq \alpha(n-1).$$

This is a contradiction with respect to equation (6). The proof of Theorem 2 follows.

□

5 Remarks and Questions

The first question which arises is how *sharp* is Theorem 2? In order to demonstrate Conjecture 1.2, one needs to study an inverse-type problem. Given the data (n, θ, I, S) in \mathbb{R}^2 , can we construct an n -dimensional symmetric convex set K such that:

$$vol_n(K) vol_n(K^\circ) = \alpha(n-1, \theta, I, S) vol_{n-1}(\mathbb{S}^{n-1})^2? \quad (9)$$

This seems plausible. Or perhaps, we could use a suitable probabilistic argument? Suppose the class of n -dimensional symmetric convex bodies is enhanced with a (suitable) probability measure. Perhaps we could show that the probability that a symmetric convex body (for which the equality (9) holds) is equal to one?

The second remark I would like to point out is about a lower bound for $\alpha(n, S, I)$, where I is a half-circle. The material for this follows from M.Fradelizi in [11]. Consulting [11], I believe in the following:

Corollary 5.1 *Let \mathbb{R}^2 be enhanced with the measure μ_2 as in definition 1.2 and let $n \geq 4$. Let K be a symmetric convex set in \mathbb{R}^2 . Then we have:*

$$\mu_2(K) \mu_2(K^\circ) \geq \frac{1}{n}.$$

Recall that an unconditional symmetric convex body in \mathbb{R}^2 is a convex body symmetric with respect to the origin of \mathbb{R}^2 , which is also symmetric with respect to the x and y -axis of the Cartesian coordinates. For unconditional symmetric convex bodies in \mathbb{R}^2 , Conjecture 5.1 holds and can be obtained from the following result of Saint-Raymond :

Theorem 3 (Saint-Raymond) *Let K be an unconditional symmetric convex body in \mathbb{R}^n . Let $K_+ = K \cap \mathbb{R}_+^n$ and $K_+^\circ = K^\circ \cap \mathbb{R}_+^n$. Then for every $(m_1, \dots, m_n) \in (0, \infty)^n$ we have:*

$$\int_{K_+} \left(\prod_{i=1}^n x_i^{m_i-1} \right) dx \int_{K_+^\circ} \left(\prod_{i=1}^n x_i^{m_i-1} \right) dx \geq \frac{1}{\Gamma(m_1 + \dots + m_{n+1})} \prod_{i=1}^n \frac{\Gamma(m_i)}{m_i}.$$

An idea to prove 5.1 is to prove that for every symmetric convex set S in \mathbb{R}^2 , there exists an unconditional symmetric convex set $U(S)$ such that:

$$\mu_2(S)\mu_2(S^\circ) \geq \mu_2(U(S))\mu_2(U(S)^\circ).$$

Assuming S be an arbitrary symmetric body in \mathbb{R}^2 , apply the Steiner symmetrisation once with respect to the x -axis, and denote this set by $U_x(S)$. Apply the Steiner symmetrisation with respect to y -axis for the set $U_x(S)$ and denote the result by $U(S)$. It is clear that $U(S)$ is an unconditional symmetric convex body in \mathbb{R}^2 and without any particular difficulty we can demonstrate that:

$$\mu_2(S) \geq \mu_2(U(S)).$$

All that remains to prove Conjecture 5.1 is to show :

$$\mu_2(S^\circ) \geq \mu_2(U(S)^\circ)(?)$$

In order to demonstrate the above inequality, it was pointed out to me in [11] to consider the technique of Shadow systems in [9], [48] and [52]. This technique has shown interest in the study of Mahler Conjecture, [14], [10], [42]. Indeed it is possible to generalise the proof of Theorem 1 in [9] for our case (where the measure has an $\frac{1}{n}$ -affine density $|x|^n$) however, unfortunately it is more likely that we obtain the inverse inequality :

$$\mu_2(S^\circ) \leq \mu_2(U(S)^\circ).$$

Indeed this still does not provide counter-examples for Conjecture 5.1, but at this point I have not studied this more carefully.

Should Conjecture 5.1 be true, we will have:

$$\alpha(n, I, S) \geq \frac{1}{(n-1)C(n)^2},$$

where

$$C(n) = \int_{-\pi/2}^{+\pi/2} \cos(t)^{n-2} dt.$$

And we will have:

$$\begin{aligned} \frac{\text{vol}_{n-1}(\mathbb{S}^{n-1})^2}{(n-1)C(n)^2} &= \frac{\text{vol}_{n-2}(\mathbb{S}^{n-2})^2}{n-1} \\ &= \frac{4\pi^{n-1}}{(n-1)\Gamma((n-1)/2)^2} \\ &> \frac{4^n}{\Gamma(n+1)}. \end{aligned}$$

which is good news with respect to the lower bound conjectured by Mahler.

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